CIT 5920–Mathematical Foundations of Computer Science
Midterm 1: *Sets, Relations, Functions, and Combinatorics*

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Exam Instructions

- **PLEASE DO NOT BEGIN UNTIL EXPLICITLY INSTRUCTED TO DO SO.**
- **Leaving Early:** To ensure everybody can take the exam in a quiet environment, we ask that you remain in the room until the end of the exam. If you finish early, you can use the remaining time to review your answers.
- **Anonymity:** To ensure fairness in grading, please refrain from writing your name on the exam. We will grade papers anonymously, but rest assured that your final grade will be recorded accurately.
- **Electronic Devices:** Please set your cellphones to "Do Not Disturb" mode. Note that the use of calculators is not permitted during this exam.
- **Simplification:** It's okay to leave your answers in an unsimplified form. For instance, you don't need to simplify "2 + 2". We're assessing your understanding, not your mental arithmetic skills.
- **Answer Space:** Ensure you provide your answers within the designated spaces. We won't be checking the back of any page.
- **Scratch Paper:** There's a blank sheet provided at the end of the exam for any rough work. Additionally, feel free to use the back of any sheet if you need more space.
- **Time Management:** Some questions might be more challenging than others. We recommend not spending too much time on any single question to ensure you attempt all of them.
- **Explanations:** Unless a question specifically states "No explanation needed", please provide a brief rationale for your answer.

Your **PennID** (the 8 **digits** in big font on your penncard): ______________________________

The PrairieLearn portion will be optional (if you do not do it, we will just weigh your written portion more to compensate), will be released after both recitations today are over, and will be due at 11:59pm on Thursday, October 12th.

GOOD LUCK!

Exercise 1 – Mathematical Notation

A. Let A, B, S be sets. Translate the following statements using the best mathematical notation:

(i) "The intersection of A and B is not empty."
(ii) "The set A contains the element 3."
(iii) "The set containing just 3 is a subset of A."
(iv) "The set B does not contain the element 3."
(v) "The set A from which we remove the element 3 is equal to the set B."
(vi) "The union of A and its complement is the universe."
(vii) "The union of A and B is the set S."
(viii) "The set A is a subset of all integers."

Solution:

(i) $A \cap B \neq \emptyset$

(ii) $3 \in A$

(iii) $\{3\} \subseteq A$

(iv) $3 \notin B$

(v) $A \setminus \{3\} = B$ (on the other hand, $A \setminus B = \{3\}$ is not acceptable; see as counter-example: $A = \{3, 4\}$ and $B = \{4, 5\}$, then $A \setminus B = \{3\}$, in other words, if B is on the right-hand side of the set difference it could contain any number of elements and the latter statement would still be true, which is not what the question is asking for)

(vi) $A \cup \overline{A} = U$ is the preferred notation in this course. ($A \cup A' = U$ and $A \cup A^C = U$ are also acceptable answers.)

(vii) $A \cup B = S$

(viii) $A \subseteq \mathbb{Z}$ (some other class of integers is also acceptable, such as $\mathbb{N}$)

B. Write the following sets, which are expressed using set builder notation, using set roster notation:

(i) $\{3m \mid m \in \mathbb{Z}$ and $10 < m < 15\}$.
(ii) $\{a \mid a \in \mathbb{Z}$ and $a^2 \in \mathbb{Z}$ and $4 \leqslant a < 5\}$.
(iii) $\{3y^2 + 12 \mid y \in \mathbb{Z}$ and $-2 < y < 3\}$.
(iv) $\{x \mid x \in \mathbb{R}$ and $x \in \mathbb{Z}_4$ and $5 \leqslant x \leqslant 23\}$ (where $\mathbb{Z}_4$ is the set of integers that are multiples of 4).

Solution:

(i) $\{33, 36, 39, 42\}$

(ii) $\{4\}$

(iii) $\{12, 15, 24\}$ (also acceptable to provide $\{12, 15, 12, 24\}$ and $\{15, 12, 15, 24\}$ or any repetitions of elements)

(iv) $\{8, 12, 16, 20\}$

C. Let $C = \{1, x\}$, what does the expression $|\mathcal{P}(C)|$ represent (explain using words)? What is the value of this expression?

Solution: This expression represents the number of elements (also called cardinality) of the powerset of C. The powerset of C is the set of all possible subsets that can be formed from elements of C. This value is equal to $2^{|C|} = 2^2 = 4$ (when counting subsets, for each element, we have to decide whether to include it or not).

D. Let $D = \{6, 7, 8\}$ and $E = \{a, b, c\}$.

(i) What is $D \times E$ (using words)?

(ii) What is the value of $|D \times E|$?

(iii) Is $\{a, 8\}$ an element of $D \times E$? Why or why not?

(iv) Is $\{a, 8\}$ an element of $E \times D$? Why or why not?

(v) Provide two examples of elements of $D \times E$.

Solution:

(i) $D \times E$ is the Cartesian product of the pairs (d, e) formed, for all possible ways to choose $d \in D$ and $e \in E$.

(ii) The size/cardinality of the Cartesian product $|D \times E| = |D| \cdot |E| = 3 \cdot 3 = 9$.

(iii) The elements of $D \times E$ are ordered pairs, i.e., $(d, e) \in D \times E$, not sets. Therefore the subset $\{a, 8\}$ does not belong to $D \times E$ because it is a subset instead of an ordered pair.

(iv) Like in the previous question, $\{a, 8\}$ is a subset, not an ordered pair, therefore it does not belong to a Cartesian product. The ordered pair $(a, 8)$, using parentheses, would belong to $E \times D$.

(v) Two examples (among 9): $(6, c) \in D \times E$ and $(8, a) \in D \times E$.

Exercise 2 – Set Identity

Show using the right relations: $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.

Solution: The main difficulty in this question is to remember to recognize that $\setminus$ refers to a "set difference", and to remember the definition of this concept: $A \setminus B$ is the set of elements which belong to A but not to B, in other words,

$x \in A \setminus B$, by definition, means that $x \in A \cap \overline{B}$. Using this information and the rules on sets at the end of the exam, we derive:

$$\begin{aligned} A \cup (B \setminus C) &= A \cup (B \cap \overline{C}) && \text{[definition of set difference]} \\ &= (A \cup B) \cap (A \cup \overline{C}) && \text{[distributivity of intersection over union]} \\ &= (A \cup B) \cap (\overline{\overline{A} \cap C}) && \text{[using De Morgan's law]} \\ &= (A \cup B) \setminus (\overline{A} \cap C) && \text{[definition of set difference]} \\ &= (A \cup B) \setminus (C \cap \overline{A}) && \text{[commutativity of intersection]} \\ &= (A \cup B) \setminus (C \setminus A) && \text{[definition of set difference]} \end{aligned}$$

Thus,

$$A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A).$$

Note that we could also make the computation from the other direction (start from the right-hand side of the equation, $(A \cup B) \setminus (C \setminus A)$, and apply rules until we get to the left-hand side).

Exercise 3 – Reflexive Relations

A student makes the following statement:

"*On a set of 3 elements, it is impossible to have a relation that has at least 2 elements such that it is symmetric and transitive but not reflexive.*"

Is the student right? If they are right, provide some brief justification. If they are wrong, provide a counter-example.

Solution: The student is incorrect.

Consider for example $\{(1,2),(2,1),(1,1),(2,2)\}$ on the set $S = \{1,2,3\}$. It is symmetric and transitive but because it does not include $(3,3)$, it is not reflexive.

The reflexive property must hold for *all* elements in the set on which the relation is defined. In this case, the set S has 3 elements, so the relation must include $(3,3)$ to be reflexive. In contrast, the symmetric and transitive properties only need to hold for the elements referenced in the relation, which do not need to involve all the elements of S.

Exercise 4 – Symmetric Relations

A. On the set $\{1,2,3\}$ is the relation $\{(1,1),(2,2),(3,3)\}$ symmetric? Just answer yes or no. No explanation needed.

Solution: This is true. There is no distinct pair so in some sense there is nothing to verify in terms of an a being related to b and whether or not b is also related to a.

B. Consider a set A that has n elements. How many distinct symmetric relations can be defined on A? Please provide an explanation.

Solution: For the symmetric relations part we just need to find the correct set of elements from which any arbitrary subset can be picked.

Note that for a symmetric relation all the pairs of the form (x, x) are totally fine. *That was the reason for the example provided in the first part of the exercise.* So we have those n pairs that we can safely pick from.

Also, we know that whenever we pick something like $(1, 2)$ we will automatically pick $(2, 1)$. Therefore, when we look at the other pairs to pick we just need to consider pairs without worrying about order. That amounts to essentially $\binom{n}{2}$.

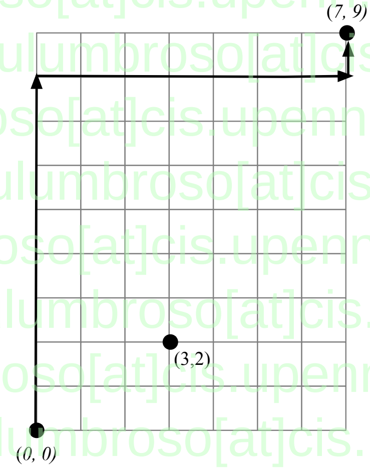
That means that we have a total of $\binom{n}{2} + n$ pairs and any subset of these pairs will give us a symmetric relation (again, if we pick an (a, b) pair we must remember to automatically add a (b, a) pair to keep it symmetric).

Therefore $2^{\binom{n}{2}+n}$.

Exercise 5 – Counting Monotonic Paths in a Grid

Consider a grid that is 7 units wide and 9 units tall. A *monotonic path* is one that begins at point $(0, 0)$ (the bottom left corner) and traverses to $(7, 9)$ (the top right corner), using only upward ($\uparrow$) and rightward ($\rightarrow$) moves. For example, the path on the right is a monotonic path:

$$(\uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \uparrow).$$



A. How many different monotonic paths are possible?

Solution: To get from point $(0, 0)$ to $(7, 9)$, one must move right 7 times and up 9 times, regardless of the order. This is equivalent to asking in how many ways we can arrange 7 rightward moves and 9 upward moves.

The number of ways to choose k items from n items (without regard to order) is given by the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In this case, we want to choose 7 rightward moves from a total of $7 + 9 = 16$ moves. So, $n = 16$ and $k = 7$:

$$\binom{7+9}{7} = \binom{16}{7} = \frac{16!}{7!9!}$$

This gives the number of monotonic paths from $(0, 0)$ to $(7, 9)$.

B. How many such monotonic paths go through the point $(3, 2)$?

Solution: To solve this, we break the problem into two parts:

1. The number of paths from $(0, 0)$ to $(3, 2)$.
2. The number of paths from $(3, 2)$ to $(7, 9)$.

For the first part, we need 3 rightward moves and 2 upward moves, a total of 5 moves. This is equivalent to choosing 3 rightward moves from 5 moves:

$$\binom{3+2}{3} = \binom{5}{3} = \frac{5!}{3!2!}$$

For the second part, from $(3, 2)$ to $(7, 9)$, we need 4 rightward moves and 7 upward moves, a total of 11 moves. This is equivalent to choosing 4 rightward moves from 11 moves:

$$\binom{4+7}{4} = \binom{11}{4} = \frac{11!}{4!7!}$$

The total number of paths that go through $(3, 2)$ is the product of these two values.

Exercise 6 – Picking Teams

A. There are three different fields for the students to practice on. How many different ways are there to assign the 27 players to the 3 fields in teams of 9—knowing that only 1 team can ?

Solution: First, we choose 9 players out of 27 for the first field, which can be done in $\binom{27}{9}$ ways. Then, we choose 9 players out of the remaining 18 for the second field, which can be done in $\binom{18}{9}$ ways. The remaining 9 players will go to the third field. So, the total number of ways is:

$$\binom{27}{9} \times \binom{18}{9} = \frac{27!}{9!18!} \times \frac{18!}{9!9!} = \frac{27!}{9!9!9!}.$$

B. How many ways are there to assign the 27 players to 3 teams of 9, without regard for which team is on which field?

Solution: This is similar to the previous question, but since the fields are indistinguishable, we divide by the number of ways to arrange the 3 teams on the fields, which is 3!. Thus, the number of ways is:

$$\frac{\binom{27}{9} \times \binom{18}{9}}{3!} = \frac{\frac{27!}{9!9!9!}}{3!} = \frac{27!}{3!9!9!9!}.$$

C. How many ways are there to assign the 27 players to 3 teams of 9, and for each team to choose 1 of its players as captain? As in the previous question, we do not care which team is on which field.

Solution: Using the result from the previous question, for each team of 9 players, there are 9 ways to choose a captain. So, the total number of ways is:

$$\frac{\frac{27!}{9!9!9!}}{3!} \times 9^3 = \frac{27! \times 9^3}{3!9!9!9!}.$$

D. One of the teams plays 10 games against teams from other schools, ending the season with a 7-3 record (they won 7 games out of the 10 games they played, and lost 3 games). How many different sequences of wins and losses could have led to this outcome?

Solution: In this question, the composition of the wins and losses is fixed, but not their order. This is equivalent to ordering 7 W's (wins) and 3 L's (losses) in a sequence. The number of ways to do this is:

$$\binom{10}{7} = \frac{10!}{7!3!}.$$

Exercise 7 – Counting Solutions

How many non-negative ($x_i \geqslant 0$) integer solutions exist for the following systems of equations or inequations (remember that a system means that all of these need to be satisfied simultaneously)?

A.

$$x_1 + x_2 + x_3 + x_4 = 21$$

Solution: To solve this, we can use the "stars and bars" method. Imagine we have 21 stars in a row and we want to divide them into 4 groups using $4 - 1 = 3$ bars. Each group represents the value of one of the variables x_1, x_2, x_3, and x_4. The number of stars in each group gives the value of the corresponding variable.

For example, the configuration:

$$*****|****|******|*****$$

represents the solution $x_1 = 5, x_2 = 4, x_3 = 6$, and $x_4 = 6$.

The problem then reduces to determining the number of ways to arrange 21 stars and 3 bars. This is equivalent to choosing 3 positions out of 24 for the bars, which is:

$$\binom{21+4-1}{4-1} = \binom{24}{3}.$$

B.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leqslant 21 \\ x_2 &\geqslant 2 \\ x_3 &\geqslant 3 \\ x_4 &\leqslant 19 \end{aligned}$$

Solution: Using the "stars and bars" concept, we first introduce a slack variable x_5 to account for the inequality:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_5 \geqslant 0$.

Given the constraints $x_2 \geqslant 2$ and $x_3 \geqslant 3$, we allocate 2 and 3 units respectively to x_2 and x_3. This modifies the equation to:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

The constraint $x_4 \leqslant 19$ is automatically satisfied as x_4 cannot exceed 19 in any valid solution.

Now, we need to find non-negative integer solutions for:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

Using "stars and bars", this is equivalent to arranging 16 stars with 4 bars, which gives:

$$\binom{20}{4}.$$

C.

$$x_1 + x_2 + x_3 + x_4 \leqslant 21$$
$$x_2 + x_3 = 5$$

Solution: Using the "stars and bars" concept, we first introduce a slack variable x_5 to account for the inequality:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

with the constraint $x_5 \geq 0$.
Given the constraint

$$x_2 + x_3 = 5, \tag{1}$$

we can rewrite the equation as:

$$x_1 + x_4 + x_5 = 16 \tag{2}$$

Now, we need to find non-negative integer solutions to this equation.
Using "stars and bars", this is equivalent to arranging 16 stars with 2 bars, which gives:

$$\binom{18}{2}$$

Finally we combine the number of ways to fulfill the constraint in (1) with the number of ways to fulfill the constraint in (2), using the product rule, to get the total number of solutions to the combined system of equations:

$$\binom{6}{1} \times \binom{18}{2}.$$

Exercise 8 – Probability
There are 15 oranges that are being thrown randomly into 6 baskets. Each basket has enough capacity to hold all 15 oranges. What is the probability that the oranges don't all fall in the same basket? For the purposes of this problem, think of the oranges as distinct and the baskets as distinct.

Solution: To solve this problem, we'll first determine the total number of ways the oranges can be thrown into the baskets, and then we'll determine the number of ways in which all the oranges can fall into the same basket. We will be using the *complement of an event*: The probability that the oranges don't all fall into the same basket will be 1 minus the probability that they all fall into the same basket.

- **Total number of ways the oranges can be thrown into the baskets:** Since there are 6 baskets and each orange can be thrown into any of the 6 baskets, the total number of ways the oranges can be thrown is 6^{15}.
- **Number of ways all the oranges fall into the same basket:** There are 6 baskets, and if all the oranges fall into one basket, there are 6 ways this can happen (one for each basket).
- **Probability that all the oranges fall into the same basket:**

$$P(\text{all in one basket}) = \frac{\text{number of ways all oranges fall into one basket}}{\text{total number of ways the oranges can be thrown}}$$
$$P(\text{all in one basket}) = \frac{6}{6^{15}}$$

- **Probability that the oranges don't all fall into the same basket:**

$$P(\text{not all in one basket}) = 1 - P(\text{all in one basket}$$
$$P(\text{not all in one basket}) = 1 - \frac{6}{6^{15}}.$$

The complement of an event in probability theory represents all outcomes that are not in the event itself. Utilizing the complement is a powerful tool, especially when it's challenging or cumbersome to directly compute the probability of a particular event. By calculating the probability of the complement (which might be simpler) and subtracting it from 1, we can indirectly find the desired probability. In the context of the oranges and baskets problem, while determining the probability of all oranges falling into one specific basket might be straightforward, calculating the probability for various other combinations can be complex. Hence, it's easier to compute the probability of the complement event "all in one basket" and subtract it from 1 to get the probability of the event "not all in one basket."

Compendium of Formulas

- The number of ways to choose k items out of n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- The number of ways to pick and arrange k items out of n is $P(n,k) = n!/k!$.
- $|P(A)| = 2^{|A|}$.
- Number of ways to arrange n items out of which k_1 are identical of one kind and k_2 are identical of another kind is.

$$\frac{n!}{k_1!k_2!}$$

Properties of Relations

- A relation is *symmetric* if and only for every a that is related to b, b is related to a.
- A relation on set X is *reflexive* if and only if for every $x \in X$, (x, x) is in the relation.
- A relation R is *transitive* if and only if aRb and bRc gives us aRc.
- A relation is antisymmetric if aRb and bRa only happens when a and b are equal.

Set Identities

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

